Boundary-Scattering Effects in the Conductivity of Thick Wires in Zero Magnetic Field and in Large Transverse Magnetic Field*

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The qualitative Dingle formula for the conductivity of thick wires in zero magnetic field is derived by the Chambers path-integral technique for arbitrary cross-sectional geometry. The result is extended to the case of a large magnetic field transverse to the wire axis. Application is made to the case of thin films in the Sondheimer geometry and to the case of circular wires. Comparison is made to experiments on K wires.

I. INTRODUCTION

Dingle[1] has studied the problem of calculating the conductivity of an infinitely long wire of circular cross section, taking into account the boundary scattering of the electrons.

As a preliminary to approaching the problem in detail, he gives a qualitative argument suggesting that for thick wires of arbitrary cross section in zero magnetic field one has

$$\sigma=\sigma_0(1-C\lambda_F P/A)\ , \qquad (1)$$

where C is a constant, λ_F is the mean free path at the Fermi surface, P is the perimeter of the wire, and A is the cross-sectional area. Dingle notes that the result of Fuchs's[2] treatment of a thin film may be put into this form when λ_F is small compared to the film thickness, and that for completely diffuse boundary scattering the constant $C=\frac{3}{16}$. Using this value of C in Eq. (1) for the case of a circular wire of radius R, Dingle suggests that one should expect

$$\sigma=\sigma_0(1-3\lambda_F/8R) \qquad (2)$$

for the case $\lambda_F \ll R$. He then proceeds to obtain expressions by the Boltzmann-equation approach which do indeed result in Eq. (2) for the thick-wire case.

This paper derives Eq. (1) utilizing the Chambers path-integral approach,[3] demonstrating that the value $C=\frac{3}{16}$ is a result of the assumptions of a spherical Fermi surface and diffuse boundary scattering. The problem is then extended to discuss the high transverse magnetic field case for wires of arbitrary cross section, and an expression quite similar to Eq. (1) is found to apply to the transverse magnetoresistivity ρ_{11}. The Hall resistivity is shown to tend to the bulk value for high magnetic field.

II. THEORY

A. $\vec{H}=0, \vec{E}$ uniform

Chambers's kinetic formulation[3] for the perturbation $f^{(1)}$ to the equilibrium Fermi distribution $f^{(0)}$ gives

$$f^{(1)}(\vec{r},\vec{v})=e\frac{df^{(0)}}{d\epsilon}\int_{t-\Delta t}^{t}\vec{E}(\vec{r}',\vec{v}',t')\cdot\vec{v}(t')\times e^{-(t-t')/\tau}dt'\ , \qquad (3)$$

where ϵ is the energy, $\vec{E}$ is the electric field, $\vec{v}$ is the velocity, τ is the relaxation time, and Δt is the largest time interval in the past that was physically possible for electrons to have diffusely scattered onto a path leading to the phase point $(\vec{r},\vec{v})$ at time $t'=t$. For bulk samples Δt is essentially infinite, while for finite samples Δt is limited by the proximity of the boundaries, if boundary scattering is diffuse. The velocity $\vec{v}(t')$ must be found by solving the classical equation of motion for an electron in the given force field (ignoring collisions), subject to the boundary conditions that at $t'=t$, $\vec{r}(t')=\vec{r}$, and $\vec{v}(t')=\vec{v}$. The solution in this case is

$$v(t')=(e\vec{E}/m)(t-t')+\vec{v}\ . \qquad (4)$$

Substitution of Eq. (4) into Eq. (3) gives two terms, one of which involves E^2. This term may be neglected if the velocity gained during the interval Δt is small compared to $\vec{v}$, the velocity of arrival at $\vec{r}$. This leads to Ohmic resistivity. Making this approximation and integrating Eq. (3) under the assumptions that τ is a constant[4] and $\vec{E}$ is uniform and time independent gives

$$f^{(1)}(\vec{r},\vec{v})=e\frac{df^{(0)}}{d\epsilon}\ \vec{E}\cdot\vec{v}\tau(1-e^{-\Delta t/\tau})\ . \qquad (5)$$

For an infinite sample Δt is infinite such that

$$f^{(1)}_{\text{bulk}}=e\frac{df^{(0)}}{d\epsilon}\ \vec{E}\cdot\vec{v}\tau\ , \qquad (6)$$

$$f^{(1)}(\vec{r},\vec{v})=(1-e^{-\Delta t/\tau})f^{(1)}_{\text{bulk}}\ . \qquad (7)$$

It is desired to calculate $f^{(1)}$ at a point in the neighborhood (within a few mean free paths) of the surface. Referring to Fig. 1, let C be the point of interest. Construct a tangent plane to the boundary at the nearest point (D) to C. Define q to be

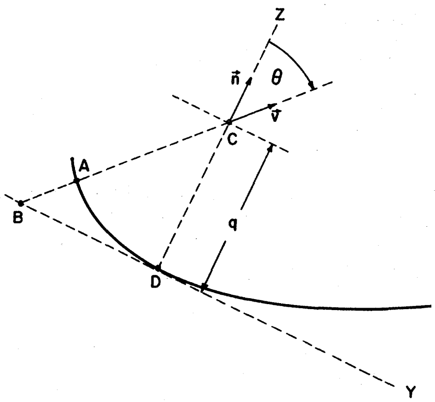

FIG. 1. Local coordinates associated with a point C at which the conductivity is to be calculated for zero magnetic field.

the perpendicular distance from the tangent plane to point C. An electron diffusely scattering from point A on the boundary travels a distance $\overline{AC}$ to arrive at C with velocity vector inclined at angle θ to the inwardly directed surface normal vector $\vec{n}$. Since the velocity gained from the electric field is assumed small compared to the velocity of arrival at C,

$$\Delta t = (\overline{AC}/v\tau)\tau = (\overline{AC}/\lambda)\tau \ . \tag{8}$$

Now assume that R_m, the minimum value of the radius of curvature of the boundary, is much larger than λ for all electrons:

$$R_m \gg \lambda_F \equiv v_F \tau \ . \tag{9}$$

This is the "thick-wire" approximation. For this case $\overline{AC}$ may be replaced by $\overline{BC}$ except when $\overline{AC}$ is large compared to λ. But for $\overline{AC} \gg \lambda$, $\Delta t \gg \tau$ so that the exponential may be neglected. Thus for thick wires

$$\Delta t = (\overline{BC}/\lambda)\tau = q\tau/\lambda\cos\theta \ . \tag{10}$$

Then for a point such as C in Fig. 1 characterized by a particular value of q

$$f^{(1)}(q, \theta \leq \tfrac{1}{2}\pi) = e(df^{(0)}/d\epsilon)\vec{E}\cdot\vec{v}\tau(1 - e^{-q/\lambda\cos\theta}) \ . \tag{11}$$

Equation (11) gives the contribution to $f^{(1)}$ from all electrons which have the associated angle $\theta \leq \frac{1}{2}\pi$. Of those electrons which arrive at C with $\theta > \frac{1}{2}\pi$, essentially all have had their last collision in the volume of the sample. These contribute $f^{(1)}_{\text{bulk}}(\theta > \frac{1}{2}\pi)$ to $f^{(1)}$ at C. Therefore,

$$f^{(1)}(q, \vec{v}) = f^{(1)}(q, \theta \leq \tfrac{1}{2}\pi)$$

or (12)

$$f^{(1)}(q, \vec{v}) = f^{(1)}_{\text{bulk}}(\theta > \tfrac{1}{2}\pi) \ .$$

Note that

$$f^{(1)}(\theta \leq \tfrac{1}{2}\pi) = f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi) - f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi)\, e^{-q/\lambda\cos\theta} \ , \tag{13}$$

such that $f^{(1)}(\theta \leq \frac{1}{2}\pi)$ vanishes for $q = 0$. Thus the Fuchs[2] boundary condition is satisfied. As indicated in Fig. 1, choose local coordinates such that z is along the inward normal, y is in the tangent plane, and x is along the length of the wire. The local current density $\vec{J}(q)$ is

$$\vec{J}(q) = (4\pi^3)^{-1} \int f^{(1)}(q, \vec{v})(-e\vec{v})\, d^3k \tag{14}$$

or

$$\begin{aligned}\vec{J}(q) = {} & (4\pi^3)^{-1} \int_{k_z \geq 0} (-e\vec{v}) f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi)\, d^3k \\ & - (4\pi^3)^{-1} \int_{k_z \geq 0} (-e\vec{v}) f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi)\, e^{-q/\lambda\cos\theta}\, d^3k \\ & + (4\pi^3)^{-1} \int_{k_z < 0} (-e\vec{v}) f^{(1)}_{\text{bulk}}(\theta > \tfrac{1}{2}\pi)\, d^3k \ .\end{aligned} \tag{15}$$

Combining the first and last terms of Eq. (15) gives

$$\vec{J}(q) = \vec{J}_{\text{bulk}} + (e^2\tau/4\pi^3) \times \int_{k_z \geq 0} \vec{v}\vec{v}\left(\frac{df^{(0)}}{d\epsilon}\right) e^{-q/v\tau\cos\theta}\, d^3k \cdot \vec{E} \ . \tag{16}$$

Taking the highly degenerate case where $df^{(0)}/d\epsilon = -\delta(\epsilon - \epsilon_F)$ and using $d^3k = (\hbar v)^{-1}\, dS(\epsilon)\, d\epsilon$, where dS is an element of area on the energy surface $\epsilon = \hbar^2 k^2/2m$, one has

$$\hat{\sigma}(q) = \hat{\sigma}^{\text{bulk}} - (e^2\tau v_F/4\pi^3\hbar) \int_{k_z \geq 0} \vec{1}_k\vec{1}_k\, e^{-q/v_F\tau\cos\theta} dS(\epsilon_F) \tag{17a}$$

or

$$\hat{\sigma}(q) = \hat{\sigma}^{\text{bulk}} - (3\sigma_0/4\pi) \int_{\theta \leq \pi/2} \vec{1}_k\vec{1}_k\, e^{-q/\lambda_F\cos\theta} \sin\theta\, d\theta\, d\phi \ , \tag{17b}$$

where

$$\sigma_0 = ne^2\tau/m \ , \tag{17c}$$

$$\vec{1}_k = \sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k} \ . \tag{17d}$$

Writing out the dyad $\vec{1}_k\vec{1}_k$ in tensor form and integrating each element over ϕ gives

$$\hat{\sigma}(q) = \hat{\sigma}^{\text{bulk}} - \frac{3\sigma_0}{4}\int_0^{\pi/2} e^{-q/\lambda_F\cos\theta} \times \begin{pmatrix} \sin^3\theta & 0 & 0 \\ 0 & \sin^3\theta & 0 \\ 0 & 0 & 2\cos^2\theta\sin\theta \end{pmatrix} d\theta \ . \tag{18}$$

Experimentally one is interested in the average of the local current density over the cross section of the wire, since one measures the cross-sectional area A and the current I flowing through it:

$$\langle\vec{J}\rangle = I/A = A^{-1}\int\int \vec{J}\, dA \ , \tag{19a}$$

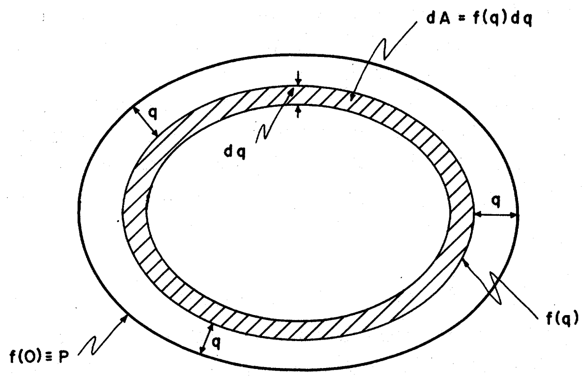

FIG. 2. Cross section of wire showing infinitesimal area generated by incrementing q to $q+dq$.

$$\langle \vec{J} \rangle = \langle \hat{\sigma} \rangle \vec{E} = A^{-1} \int\int \hat{\sigma}\, dA\, \vec{E}\ , \qquad (19b)$$

$$\langle \hat{\sigma} \rangle = A^{-1} \int\int \hat{\sigma}\, dA\ . \qquad (19c)$$

To carry out the above areal averaging of $\hat{\sigma}$ it is necessary to express $\hat{\sigma}(q)$ at each point q in a common laboratory system of coordinates. As one moves around the perimeter of the wire, the local coordinates, as defined in Fig. 1, rotate about the x axis such that on transforming each $\hat{\sigma}(q)$ to a lab system chosen with its x axis down the length of the wire, $\sigma_{11}(q)$ is unchanged. Therefore $\langle \sigma_{11} \rangle$ may be computed by using Eqs. (18) and (19c), directly. Also since the result of transforming a diagonal tensor such as Eq. (19) by a rotation about x leaves the tensor x diagonal, one has

$$\rho_{11} = \langle \sigma_{11} \rangle^{-1}\ , \qquad (20)$$

where ρ_{11} is the experimentally measured longitudinal resistivity, and

$$\langle \sigma_{11} \rangle = \sigma_{11}^{\text{bulk}} - \frac{3\sigma_0}{4} \int_0^{\pi/2} \langle e^{-q/\lambda_F \cos\theta} \rangle \sin^3\theta\, d\theta\ . \qquad (21)$$

Consider contours of constant q as indicated in Fig. 2. The differential area dA may be written as

$$dA = f(q)\, dq\ , \qquad (22)$$

where $f(q)$ is seen to be the perimeter of the contour characterized by a value of q. For regions where $q \ll R_m$, $f(q)$ is slowly varying and may be replaced by $f(0)$. Then one has

$$\langle e^{-q/\lambda_F \cos\theta} \rangle = A^{-1} f(0) \int_0^{R_m} e^{-q/\lambda_F \cos\theta}\, dq\ , \qquad (23)$$

where, since $\lambda_F \ll R_m$, the integrand is essentially zero for $q \geq R_m$. Integration gives as an excellent approximation

$$\langle e^{-q/\lambda_F \cos\theta} \rangle = \lambda_F P \cos\theta / A\ . \qquad (24)$$

Substitution of Eq. (24) into Eq. (21) gives, after integration over θ,

$$\sigma_{11} = \langle \sigma_{11} \rangle = (1 - 3\lambda_F P/16A)\sigma_0 \qquad (25)$$

or

$$\rho_{11} = (1 + 3\lambda_F P/16A)\rho_0\ , \qquad (26)$$

where $\rho_0 = 1/\sigma_0$, and Eq. (26) is obtained by taking the leading term in a binomial expansion of the inverse of Eq. (25). Equation (25) is the result Dingle[1] suggested by qualitative argument and which he derived for the case of a thick circular wire.

B. $\vec{H}$ Transverse, $\vec{E}$ Uniform

Consider the situation where the thick wire is placed in a uniform magnetic field directed perpendicularly to the axis of the wire. The velocity $\vec{v}(t')$ is to be found by solving

$$\frac{m\, d\vec{v}(t')}{dt'} = -e\vec{E} - \frac{e}{c}\, \vec{v} \times \vec{H}\ . \qquad (27)$$

The Ohmic approximation is obtained by neglecting the term $-e\vec{E}$ in Eq. (27) in order to avoid quadratic dependence on E of the integrand of Eq. (3). The solution may be written

$$\vec{v}(t') = e^{\hat{\omega}(t'-t)}\vec{v}\ , \qquad (28a)$$

$$\hat{\omega} = \frac{e\hat{H}}{mc} = \frac{e}{mc} \begin{pmatrix} 0 & -H_3 & H_2 \\ H_3 & 0 & -H_1 \\ -H_2 & H_1 & 0 \end{pmatrix}\ . \qquad (28b)$$

Then substitution of Eq. (28a) into Eq. (3) gives

$$f^{(1)}(\vec{r}, \vec{v}) = e \frac{df^{(0)}}{d\epsilon} \int_{t-\Delta t}^{t} e^{(\hat{\omega} + 1/\tau)(t'-t)}\, dt'\, \vec{v} \cdot \vec{E}\ , \qquad (29)$$

$$f^{(1)}(\vec{r}, \vec{v}) = e\left(\frac{df^{(0)}}{d\epsilon}\right)(\hat{\omega} + \tau^{-1})^{-1} \times [1 - e^{-(\hat{\omega} + \tau^{-1})\Delta t}]\, \vec{v} \cdot \vec{E}\ . \qquad (30)$$

For infinite samples Δt goes to infinity and therefore

$$f^{(1)}_{\text{bulk}} = e\left(\frac{df^{(0)}}{d\epsilon}\right)(\hat{\omega} + \tau^{-1})^{-1}\, \vec{v} \cdot \vec{E} \qquad (31)$$

and

$$f^{(1)}(\vec{r}, \vec{v}) = [1 - e^{-(\hat{\omega} + \tau^{-1})\Delta t}]\, f^{(1)}_{\text{bulk}}\ . \qquad (32)$$

Now restrict the problem to deal only with the case when H is large enough that the cyclotron radius $R_c \ll \lambda_F$. Referring to Fig. 3 consider an electron arriving at point C after scattering from the surface at point A. Draw a line from C to the boundary parallel to $\vec{H}$, and define the distance $\overline{CD} \equiv q$. The transit time Δt from A to C satisfies

$$v \cos\theta\, \Delta t - q \leq 2R_c \tan\psi\ , \qquad (33)$$

such that as H tends toward infinity one has

$$\Delta t = q/v\cos\theta\ , \tag{34}$$

except for a negligible region near limiting points of the boundary where H is tangent to the wire ($\psi \approx \frac{1}{2}\pi$). The above discussion resulting in Eq. (34) is valid only for electrons with $\cos\theta \geq 0$, that is, $\theta \leq \frac{1}{2}\pi$. Then within this high-field approximation

$$f^{(1)}(q, \theta \leq \tfrac{1}{2}\pi) = (1 - e^{-(\hat{\omega} + \tau^{-1})q/v\cos\theta}) f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi)\ . \tag{35}$$

Note that $f^{(1)}(q, \theta \leq \frac{1}{2}\pi)$ vanishes for $q = 0$ so that the Fuchs[2] boundary conditions are satisfied. Electrons arriving at a point q with $\theta > \frac{1}{2}\pi$ have either had their last collision in the volume of the sample, or are on reentrant trajectories originating at some point on the boundary. Since it is assumed that $R_m \gg \lambda$, reentrant trajectories are probable only near the limiting points of the boundary where $\psi \approx \frac{1}{2}\pi$. These contributions become negligible as H tends toward infinity. Then for large H

$$f^{(1)}(q, \vec{v}) = f^{(1)}(q, \theta \leq \tfrac{1}{2}\pi)$$

or

$$f^{(1)}(q, \vec{v}) = f^{(1)}_{\text{bulk}}(\theta > \tfrac{1}{2}\pi)\ . \tag{36}$$

Choose local coordinates such that q is measured inward along the z axis parallel or antiparallel to $\vec{H}$ (ψ is thus restricted $-\frac{1}{2}\pi \leq \psi \leq \frac{1}{2}\pi$). The local current density is calculated as

$$\vec{J}(q) = (4\pi^3)^{-1} \int_{k_z \geq 0} f^{(1)}_{\text{bulk}}(\theta \leq \tfrac{1}{2}\pi)(-e\vec{v})\, d^3k - (4\pi^3)^{-1} \int_{k_z \geq 0} \left[\left(-e\vec{v}\frac{df^{(0)}}{d\epsilon}\right)(\hat{\omega} + \tau^{-1})^{-1} \times e^{-(\hat{\omega} + \tau^{-1})q/v\cos\theta}\,\vec{v}\right] d^3k \cdot \vec{E} + (4\pi^3)^{-1} \int_{k_z < 0} (-e\vec{v}) f^{(1)}_{\text{bulk}}(\theta > \tfrac{1}{2}\pi)\, d^3k\ . \tag{37}$$

Combining the first and last integrals above gives

$$\vec{J}(q) = \vec{J}_{\text{bulk}} - \frac{e^2}{4\pi^3} \int_{k_z \geq 0} \left[\left(\frac{df_0}{d\epsilon}\right)\vec{v}(\hat{\omega} + \tau^{-1})^{-1} \times e^{-(\hat{\omega} + \tau^{-1})q/v\cos\theta}\,\vec{v}\right] d^3k \cdot \vec{E}\ . \tag{38}$$

Using $df_0/d\epsilon = -\delta(\epsilon - \epsilon_F)$, $d^3k = (\hbar v)^{-1} dS(\epsilon)\, d\epsilon$, and the identity $\vec{a}(\hat{Q}_1\hat{Q}_2\vec{b}) = [\vec{a}(\hat{Q}_2\vec{b})]\tilde{Q}_1$, where tilde means "transpose," Eq. (38) becomes

$$\hat{\sigma}(q) = \hat{\sigma}_{\text{bulk}} - \frac{3}{4\pi} \int_{k_z > 0} \vec{1}_k\, e^{-(\hat{\omega} + \tau^{-1})q/v_F\cos\theta}\,\vec{1}_k \sin\theta\, d\theta\, d\phi\, \hat{\sigma}_{\text{bulk}}\ , \tag{39}$$

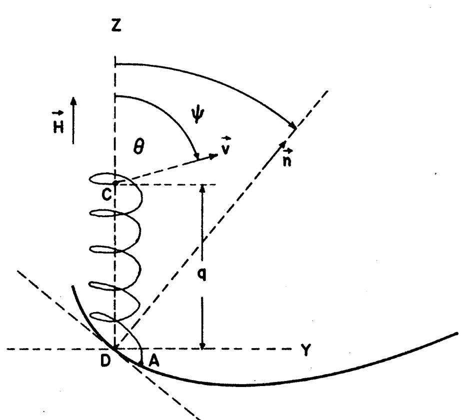

FIG. 3. Local coordinates associated with a point C at which the conductivity is to be calculated for large transverse magnetic field.

where

$$\hat{\sigma}_{\text{bulk}} = \sigma_0 \begin{pmatrix} \dfrac{1}{1 + (\omega\tau)^2} & -\dfrac{\omega\tau}{1 + (\omega\tau)^2} \\ \dfrac{\omega\tau}{1 + (\omega\tau)^2} & \dfrac{1}{1 + (\omega\tau)^2} \end{pmatrix} \tag{40a}$$

or

$$\hat{\sigma}_{\text{bulk}} = \sigma_0 (1 - \hat{\omega}\tau)^{-1} \tag{40b}$$

in a two-dimensional representation. In a three-dimensional representation all other elements are zero with the exception of $\sigma^{\text{bulk}}_{33}$ which is of no interest here. Equation (40a) gives $\hat{\sigma}_{\text{bulk}}$ for all field strengths, but is of this form only for $\vec{H}$ along the $+z$ axis. [Equation (40b) is valid for arbitrary coordinates.] The tensor $\hat{\omega} + \tau^{-1}$ has eigenvalues and eigenvectors

$$X_1 = \tau^{-1} - i\omega\ , \quad |X_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{41a}$$

$$X_2 = \tau^{-1} + i\omega\ , \quad |X_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\ . \tag{41b}$$

The general expression for a function of a matrix $g(\vec{M}) = \sum |X_k\rangle g(X_k) \langle X_k|$ gives

$$e^{\alpha(\hat{\omega} + \tau^{-1})} = e^{\alpha/\tau} \begin{pmatrix} \cos\alpha\omega & -\sin\alpha\omega \\ \sin\alpha\omega & \cos\alpha\omega \end{pmatrix}\ , \tag{42}$$

where $\alpha = -q/v_F\cos\theta$. Using Eq. (42) to write

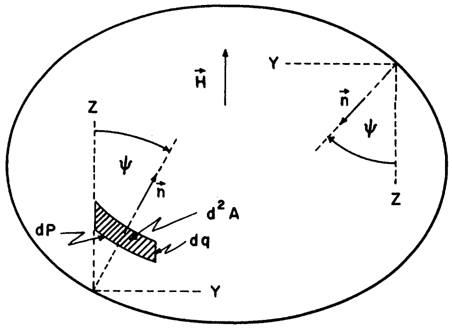

FIG. 4. Local coordinates for regions where $\vec{n}\cdot\vec{H}\geq 0$ and $\vec{n}\cdot\vec{H}\leq 0$.

out the dyad in the integrand of Eq. (39) and integrating over ϕ gives

$$\hat{\sigma}(q)=\hat{\sigma}^{\text{bulk}}-\frac{3}{4}\int_0^{\pi/2} e^{-q/\lambda_F\cos\theta} \times\begin{pmatrix}\cos\alpha\omega & \sin\alpha\omega\\ -\sin\alpha\omega & \cos\alpha\omega\end{pmatrix}\sin^3\theta\, d\theta\ . \qquad (43)$$

It is now necessary to average Eq. (43) over the cross section of the wire. Consider Fig. 4. Divide the perimeter of the wire into regions where $\vec{n}\cdot\vec{H}\geq 0$ and $\vec{n}\cdot\vec{H}\leq 0$. Equation (43) is directly applicable to all points having associated inward normal such that $\vec{n}\cdot\vec{H}\geq 0$, since $\vec{H}$ was chosen along the z axis, but for points for which $\vec{n}\cdot\vec{H}<0$ the z axis is antiparallel to $\vec{H}$. Equation (43) gives the correct result for such a coordinate system if the sign of ω is changed. However, the coordinate system in which this new $\hat{\sigma}(q)$ is valid must be rotated 180° about the x axis before averaging is carried out. This changing of the sign of ω in Eq. (43) followed by a rotation of 180° merely gives Eq. (43) again. Therefore, one may proceed to average the integrand of Eq. (43), and the result is valid for the coordinate system with $\vec{H}$ directed along the $+z$ axis. Consider

$$\langle e^{-q/\lambda_F\cos\theta}\cos\alpha\omega\rangle=A^{-1}\int\int e^{-q/\lambda_F\cos\theta}\cos\alpha\omega\, d^2A\ , \qquad (44a)$$

$$d^2A=dP(q)\,dq\cos\psi(q)\ , \qquad (44b)$$

where, as indicated in Fig. 4, $dP(q)$ is an element of perimeter of a $q=$ constant contour. Define

$$\int\cos\psi(q)\,dP(q)=P(q)\langle\cos\psi(q)\rangle\ , \qquad (45)$$

so that $\langle\cos\psi(q)\rangle$ is a perimeter average of $\cos\psi$ around a particular q contour. Note that ψ has been previously restricted to $-\frac{1}{2}\pi\leq\psi\leq\frac{1}{2}\pi$. Then one has

$$\langle e^{-q/\lambda_F\cos\theta}\cos\alpha\omega\rangle=A^{-1}\int_0^{R_m}P(q) \times\langle\cos\psi(q)\rangle e^{-q/\lambda_F\cos\theta}\cos(\alpha\omega)\,dq\ . \qquad (46)$$

Since for $\lambda_F\ll R_m$, $P(q)$ and $\langle\cos\psi(q)\rangle$ are slowly varying functions of q, one may factor them outside of the integral with their $q=0$ values and replace R_m by ∞, yielding

$$\langle e^{-q/\lambda_F\cos\theta}\cos\alpha\omega\rangle=P(0) \times\langle\cos\psi(0)\rangle\lambda_F\cos\theta/A(\omega\tau)^2\ , \qquad (47)$$

where $\omega\tau\gg 1$ has been used. Similarly

$$\langle e^{-q/\lambda_F\cos\theta}\sin\alpha\omega\rangle=-P(0)\langle\cos\psi(0)\rangle\lambda_F\cos\theta/A\omega\tau\ . \qquad (48)$$

Then one has, after integration of the averaged Eq. (43) over θ,

$$\hat{\sigma}=\langle\hat{\sigma}(q)\rangle=\hat{\sigma}_{\text{bulk}}-\frac{3P(0)\langle\cos\psi(0)\rangle\lambda_F}{16A(\omega\tau)^2} \times\begin{pmatrix}1 & -\omega\tau\\ \omega\tau & 1\end{pmatrix}\hat{\sigma}_{\text{bulk}}\ . \qquad (49)$$

In the high-field limit $\omega\tau\gg 1$, Eq. (40a) becomes

$$\hat{\sigma}_{\text{bulk}}=\frac{\sigma_0}{(\omega\tau)^2}\begin{pmatrix}1 & -\omega\tau\\ \omega\tau & 1\end{pmatrix}\ . \qquad (50)$$

Using Eq. (50) in Eq. (49) and keeping lowest-order terms in $(\omega\tau)^{-1}$ gives

$$\sigma_{11}=(1+\delta)\sigma_0/(\omega\tau)^2=(1+\delta)\sigma_{11}^{\text{bulk}}\ , \qquad (51a)$$

$$\sigma_{12}=\sigma_{12}^{\text{bulk}}=-\sigma_0/\omega\tau\ , \qquad (51b)$$

$$\rho_{11}=(1+\delta)\rho_{11}^{\text{bulk}}=(1+\delta)\rho_0\ , \qquad (51c)$$

$$\rho_{21}=\rho_{21}^{\text{bulk}}=-\rho_0\omega\tau\ , \qquad (51d)$$

$$\delta=3\lambda_F P\langle\cos\psi\rangle/16A\ . \qquad (51e)$$

III. DISCUSSION

Equations (25) and (26) give the results obtained in the zero-magnetic-field case in agreement with Dingle's[1] argument. This result may be interpreted qualitatively as indicated in Fig. 5. Although there is an exponential change in the conductivity as one moves from the boundary into the volume of the material, imagine the area A divided into a skin

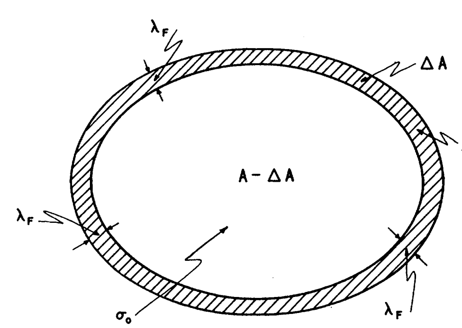

FIG. 5. In zero magnetic field one may visualize a skin region of depth λ_F in which the conductivity is reduced below the bulk value associated with the interior of the wire.

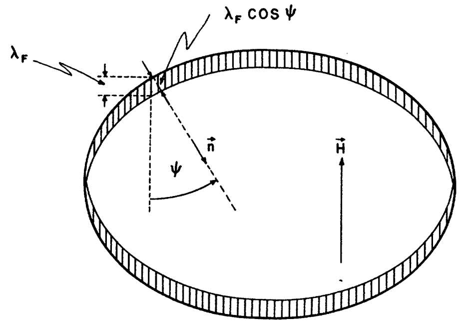

FIG. 6. In large transverse magnetic field the skin region is of depth λ_F measured parallel to the magnetic field lines.

region of depth λ_F and conductivity $\sigma_0 - \Delta\sigma$, while the interior region of area $A - \Delta A$ is characterized by σ_0. Then, since $\Delta A = \lambda_F P$ for $\lambda_F \ll R_m$ (thick wires),

$$\langle\sigma\rangle = \frac{\sigma_0(A - \Delta A) + (\sigma_0 - \Delta\sigma)\Delta A}{A} \quad (52a)$$

$$= \sigma_0\left(1 - \frac{\Delta\sigma}{\sigma_0}\frac{\lambda_F P}{A}\right) , \quad (52b)$$

where one might expect $\Delta\sigma/\sigma_0$ to be characteristic of the Fermi surface and type of boundary scattering. This is essentially Dingle's[1] argument. Note, however, that the conductivity is a symmetric tensor with the length of the wire as one principal axis, and the other principal directions are dictated by the specific geometry under consideration. As indicated in Sec. II, only σ_{11} may be found by the Dingle method. Figure 6 may be used to interpret Eq. (51) for the transverse-magnetic-field case in a similar manner. Here one has a skin of variable depth. At any point on the surface, the skin depth measured parallel (or antiparallel) to $\vec{H}$ is λ_F, but measured along the inward normal line it is $\lambda_F \cos\psi$, where ψ is the angle $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$ between the inward surface normal $\vec{n}$ and the magnetic field lines.

For the case of a circular wire of radius R

$$P/A = 2/R , \quad \langle\cos\psi\rangle = 2/\pi , \quad (53)$$

such that at zero magnetic field

$$\sigma_{11} = \sigma_0(1 - 3\lambda_F/8R) \qquad (H = 0) , \quad (54)$$

while at high field Eqs. (51) give

$$\sigma_{11} = \sigma_0 \frac{1 + 3\lambda_F/4\pi R}{(\omega\tau)^2} \qquad (\omega\tau \gg 1) , \quad (55a)$$

$$\rho_{11} = \rho_0(1 + 3\lambda_F/4\pi R) \qquad (\omega\tau \gg 1) , \quad (55b)$$

$$\rho_{21} = \rho_{21}^{\text{bulk}} = -\rho_0\omega\tau \qquad (\omega\tau \gg 1) . \quad (55c)$$

Mackey, Deering, and Sybert[5] have treated the case of a circular wire in a transverse magnetic field for $\omega\tau \gg 1$. In that paper emphasis was upon the oscillatory phenomena which occur for $\lambda_F \gtrsim R$. Their high-field expressions may be expanded for the case $\lambda_F \ll R$ to obtain precisely the results given above. Sondheimer[6] has treated the case of an infinite thin film with magnetic field perpendicular to the plane of the film. He finds that for thick films the zero-field and high-field saturation values of ρ_{11} are identical, contrary to the case of the circular wire described above. Comparison of Eqs. (26) and (51) show why this is to be expected: $\langle\cos\psi\rangle = 1$ for this exceptional geometry.

Babisken and Siebenmann[7] have recently studied saturation effects in the magnetoresistance of potassium. Using a 1.0-mm-diam wire at 1.4 °K, they have observed the decrease in ρ_{11} between $H = 0$ and $H = \infty$ (their saturation occurs at about 5 kG) due to boundary-scattering effects. This decrease agrees with that predicted by Eqs. (26) and (55b) to about 5%.

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[1]R. B. Dingle, Proc. Roy. Soc. (London) A201, 545 (1950).

[2]K. Fuchs, Proc. Cambridge Phil. Soc. 34, 100 (1938).

[3]R. G. Chambers, Proc. Roy. Soc. (London) A202, 378 (1950).

[4]If one chooses to consider $\tau = \tau(\epsilon)$ all results remain the same, if one interprets $\tau = \tau(\epsilon_F)$, since the highly degenerate approximation is utilized in this paper.

[5]H. J. Mackey, W. D. Deering, and J. R. Sybert, Phys. Rev. 176, 857 (1968).

[6]E. H. Sondheimer, Phys. Rev. 80, 401 (1950).

[7]J. Babiskin and P. G. Siebenmann (private communication).